

# Quantum unitary evolution of linearly polarized $\mathbb{S}^1 \times \mathbb{S}^2$ and $\mathbb{S}^3$ Gowdy models coupled to massless scalar fields

J. Fernando Barbero G.\* and Daniel Gómez Vergel†

*Instituto de Estructura de la Materia,  
CSIC, Serrano 123, 28006 Madrid, Spain*

Eduardo J. S. Villaseñor‡

*Grupo de Modelización y Simulación Numérica, Universidad Carlos III de Madrid,  
Avda. de la Universidad 30, 28911 Leganés, Spain and*

*Instituto de Estructura de la Materia,  
CSIC, Serrano 123, 28006 Madrid, Spain*

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The purpose of this paper is to study in detail the problem of defining unitary evolution for linearly polarized  $\mathbb{S}^1 \times \mathbb{S}^2$  and  $\mathbb{S}^3$  Gowdy models (in vacuum or coupled to massless scalar fields). We show that in the Fock quantizations of these systems no choice of acceptable complex structure leads to a unitary evolution for the original variables. Nonetheless, unitarity can be recovered by suitable redefinitions of the basic fields. These are dictated by the time dependent conformal factors that appear in the description of the standard deparameterized form of these models as field theories in certain curved backgrounds. We also show the unitary equivalence of the Fock quantizations obtained from the  $SO(3)$ -symmetric complex structures for which the dynamics is unitarily implemented.

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## I. INTRODUCTION

Gowdy models [1, 2] are interesting  $U(1) \times U(1)$  symmetry reductions of  $(1+3)$ -general relativity that have been used for a number of years as test beds for quantum gravitational techniques [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. They are receiving a lot of attention these days as the next arena to test loop quantum gravity. In this sense they are the natural continuation of the minisuperspace reductions considered so far in loop quantum cosmology. The fact that they can be exactly solved classically and admit a simple enough Hamiltonian description after deparameterization (see [17] for a rigorous classical treatment of these models coupled to matter scalar fields) makes them very attractive from this point of view.

One of the most striking features of the Fock space quantization for the Gowdy  $\mathbb{T}^3$  model

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\*fbarbero@iem.cfmac.csic.es

†dgvergel@iem.cfmac.csic.es

‡ejsanche@math.uc3m.es

is the impossibility of defining a unitary quantum evolution operator when the system is written in the natural field variables usually employed to describe it as a  $(1+2)$ -dimensional field theory [10, 11]. Nevertheless this is not an unsurmountable problem because it is possible to introduce a time-dependent field redefinition that leads to a unique (up to unitary equivalence) quantization, with unitary time evolution, when one demands invariance under the residual  $U(1)$  symmetry [13, 14, 16].

The purpose of this paper is to study the problem of the unitary implementation of dynamics as a natural extension of the previous literature devoted to the vacuum Gowdy  $\mathbb{T}^3$  model. The proposed generalization is two-fold. On one hand we will deal with the remaining topologies admissible for the compact Gowdy models, i.e. the three-handle  $\mathbb{S}^1 \times \mathbb{S}^2$  and the three-sphere  $\mathbb{S}^3$ . On the other hand we will consider the addition to the models of certain matter fields –massless scalars– symmetric under spatial isometries [18]. Here we will closely rely on the results of [17].

Our starting point is the interpretation of the compact Gowdy models in the different topologies as scalar field theories in very specific curved backgrounds. As shown in [17] all these models can be reinterpreted as given by the evolution of massless scalar fields in some geometric backgrounds that are conformally equivalent to the simplest metrics that can be defined on each of the relevant  $(1+2)$ -dimensional space-time manifolds. In particular, for the  $\mathbb{T}^3$  case, the metric is just the flat metric on  $(0, \infty) \times \mathbb{T}^2$  whereas in the  $\mathbb{S}^1 \times \mathbb{S}^2$  and  $\mathbb{S}^3$  examples the metric is the Einstein metric<sup>1</sup> on  $(0, \pi) \times \mathbb{S}^2$ . The corresponding conformal factors are simple functions of  $t$  ( $t$  and  $\sin t$  respectively). We will use here this description to gain useful insights on the problem of the unitary implementability of quantum time evolution.

As a first step towards this goal we show, by a direct argument, that no choice of  $SO(3)$ -invariant complex structure leads to unitary quantum evolution in terms of the variables in which these systems are naturally written<sup>2</sup>. The way out of this seemingly unavoidable obstruction to quantization is to introduce a time-dependent field redefinition as in [13]; in fact, by a simple re-scaling of the scalar fields involving *precisely* the conformal factors mentioned above we can get a well defined and unitary quantum evolution not only in the  $\mathbb{T}^3$  model but for the other topologies as well. A way to understand what is going on is to realize that the singular behavior introduced by the conformal factors is translated, in terms of the redefined fields, into the behavior of a singular, time-dependent, potential term for the re-scaled fields. Time evolution can now be implemented unitarily as a direct consequence of the fact that, in spite of being singular at some instants of time, these potentials are sufficiently well behaved as functions of the time variable in a definite sense that will be explained below. We also show the uniqueness –modulo unitary equivalence– of the Fock quantizations that allow the unitary implementation of the dynamics.

The paper is organized as follows. After this introduction we will study in section II the canonical and covariant phase space descriptions of  $\mathbb{S}^1 \times \mathbb{S}^2$  and  $\mathbb{S}^3$  Gowdy models coupled to massless scalar fields, as well as their classical dynamics. This is done by writing the field

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<sup>1</sup> In the  $\mathbb{S}^3$  case this is a non-trivial result [17] that is found by carefully considering the relevant regularity condition for the fields.

<sup>2</sup> In the vacuum Gowdy  $\mathbb{T}^3$  model this result is a direct corollary of the uniqueness theorem appearing in [16]. Here we will concentrate on the remaining topologies. Similar results for Gowdy  $\mathbb{T}^3$  coupled to massless scalar fields can be derived by our methods in a straightforward way.

equations with the help of a certain background metric. We also discuss the appropriate mode decomposition of the fields. Section III is devoted to several issues related to the Fock quantizations of these systems. In particular, we obtain different (in general unitarily nonequivalent) Fock representations for the canonical commutation relations characterized by a two-parameter family of  $SO(3)$ -invariant complex structures. Section IV deals specifically with the discussion of unitarity for the topologies considered in the paper. Whereas it is not possible to implement in a unitary way the linear symplectic transformation associated to the time evolution for the original variables, we show that a suitable re-scaling of the fields dictated by the conformal factor of the background metric leads to unitarity. This quantization is unique up to unitary equivalence. We also show here that despite having a well-defined and unitary quantum dynamics the action of the Hamiltonian operator is not defined on the Fock vacuum. This result is analogous to the one found by the authors of [13] in the  $\mathbb{T}^3$  case. We end the paper in section V with several comments and a discussion of the results.

## II. REDUCED PHASE SPACE AND CLASSICAL DYNAMICS

We discuss in this section the reduced phase space and the classical evolution for the Gowdy models –in vacuum or coupled to massless scalars– corresponding to the  $\mathbb{S}^1 \times \mathbb{S}^2$  and  $\mathbb{S}^3$  topologies. The dynamics of the local degrees of freedom that parameterize the reduced phase space in both cases [17] can be described by the same simple field equations<sup>3</sup>. These can be written as wave equations with the help of a certain auxiliary globally hyperbolic space-time background  $((0, \pi) \times \mathbb{S}^2, \mathring{g}_{ab})$  and an extra symmetry condition: invariance under the diffeomorphisms generated by a Killing vector field  $\sigma^a$  of  $\mathring{g}_{ab}$ . Explicitly the metric  $\mathring{g}_{ab}$  is

$$\mathring{g}_{ab} = \sin^2 t [-(dt)_a(dt)_b + \gamma_{ab}], \quad (2.1)$$

where  $\gamma_{ab}$  is the round unit metric on the 2-sphere  $\mathbb{S}^2$ . Using spherical coordinates  $(\theta, \sigma) \in (0, \pi) \times (0, 2\pi)$  on  $\mathbb{S}^2$

$$\mathring{g}_{ab} = \sin^2 t [-(dt)_a(dt)_b + (d\theta)_a(d\theta)_b + \sin^2 \theta (d\sigma)_a(d\sigma)_b],$$

and the Killing field  $\sigma^a$  is simply  $(\partial/\partial\sigma)^a$ .

The field equations can be derived, by imposing the additional symmetry condition  $\mathcal{L}_\sigma \phi_i = 0$  ( $i = 0, \dots, N$ ) on the solutions, from the action

$$\begin{aligned} S(\phi_i) &= -\frac{1}{2} \sum_{i=0}^N \int_{[t_0, t_1] \times \mathbb{S}^2} |\mathring{g}|^{1/2} \mathring{g}^{ab} (d\phi_i)_a (d\phi_i)_b \\ &= \frac{1}{2} \sum_{i=0}^N \int_{t_0}^{t_1} dt \int_{\mathbb{S}^2} |\gamma|^{1/2} \sin t \left( \dot{\phi}_i^2 + \phi_i \Delta_{\mathbb{S}^2} \phi_i \right). \end{aligned} \quad (2.2)$$

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<sup>3</sup> In the following we will not consider the dynamics of the global modes because it is irrelevant to the quantum unitarity issues that we want to discuss in the paper. They can be quantized in a straightforward way in terms of standard position and momentum operators with dense domain in  $L^2(\mathbb{R})$ .

Here and in the following  $\dot{\phi} := \partial\phi/\partial t$ ,  $\Delta_{\mathbb{S}^2}$  is the Laplace-Beltrami operator on the round sphere  $\mathbb{S}^2$ , and  $\mathcal{L}$  denotes the Lie derivative. As shown in [17] one of the scalar fields, say  $\phi_0$ , encodes the local gravitational degrees of freedom and the remaining ones,  $\phi_i$ ,  $i = 1, \dots, N$ , describe the matter modes added to the Gowdy models. As we can see they completely decouple in this description<sup>4</sup> because, at variance with the  $\mathbb{T}^3$  case, no extra constraint remains. Owing to this fact we omit in the following the  $i$  index whenever it is not necessary to explicitly separate gravitational and matter modes.

### A. Canonical and covariant phase spaces

We start by explicitly writing the linear space of smooth and symmetric real solutions to the massless Klein-Gordon equation of motion as

$$\begin{aligned} \mathcal{S} &:= \{ \phi \in C^\infty((0, \pi) \times \mathbb{S}^2; \mathbb{R}) \mid \dot{g}^{ab} \overset{\circ}{\nabla}_a \overset{\circ}{\nabla}_b \phi = 0; \mathcal{L}_\sigma \phi = 0 \} \\ &= \{ \phi \in C^\infty((0, \pi) \times \mathbb{S}^2; \mathbb{R}) \mid \ddot{\phi} + \cot t \dot{\phi} - \Delta_{\mathbb{S}^2} \phi = 0; \mathcal{L}_\sigma \phi = 0 \} \end{aligned} \quad (2.3)$$

endowed with the (weakly) symplectic structure  $\Omega$  induced by (2.2)

$$\Omega(\phi_1, \phi_2) := \sin t \int_{\mathbb{S}^2} |\gamma|^{1/2} \iota_t^* (\phi_2 \dot{\phi}_1 - \phi_1 \dot{\phi}_2). \quad (2.4)$$

Here  $\iota_t : \mathbb{S}^2 \rightarrow (0, \pi) \times \mathbb{S}^2$  denotes the inclusion given by  $\iota_t(s) = (t, s) \in (0, \pi) \times \mathbb{S}^2$ . It is straightforward to show that  $\Omega$  does not depend on  $t$ . We will refer to the symplectic space  $\Gamma := (\mathcal{S}, \Omega)$  as the *covariant phase space* of the system.

On the other hand, we will denote the *canonical phase space* as  $\Upsilon := (\mathbf{P}, \omega)$ . This is the space of smooth and symmetric<sup>5</sup> Cauchy data  $(Q, P) \in \mathbf{P}$  endowed with the standard symplectic structure

$$\omega((Q_1, P_1), (Q_2, P_2)) := \int_{\mathbb{S}^2} |\gamma|^{1/2} (Q_2 P_1 - Q_1 P_2). \quad (2.5)$$

Given any value of  $t$ , the bijection  $\mathfrak{I}_t : \Upsilon \rightarrow \Gamma$ , that maps every Cauchy data  $(Q, P)$  to the unique solution  $\phi \in \mathcal{S}$  such that  $\phi(t, s) = Q(s)$  and  $(\sin t) \dot{\phi}(t, s) = P(s)$ , is a linear symplectomorphism  $\omega = \mathfrak{I}_t^* \Omega$ .

Elements in the linear space  $\mathcal{S}$  can be expanded as<sup>6</sup>

$$\phi(t, s) = \sum_{\ell=0}^{\infty} \left( a_\ell y_\ell(t) Y_{\ell 0}(s) + \overline{a_\ell y_\ell(t) Y_{\ell 0}(s)} \right), \quad (2.6)$$

where  $Y_{\ell 0}$  denote the spherical harmonics that, in the standard spherical coordinates, have

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<sup>4</sup> Notice that, in spite of the apparent simplicity of the reduced phase space description, the full  $(3+1)$ -dimensional metric that solves the Einstein-Klein-Gordon equations depends both on the gravitational and scalar modes in a non-trivial way [17].

<sup>5</sup> Using set theoretical language  $\mathbf{P} := \{(Q, P) \in C^\infty(\mathbb{S}^2; \mathbb{R}) \times C^\infty(\mathbb{S}^2; \mathbb{R}) \mid \mathcal{L}_\sigma Q = \mathcal{L}_\sigma P = 0\}$ .

<sup>6</sup> The bar denotes complex conjugation.

the form

$$Y_{\ell 0}(s) = \left( \frac{2\ell + 1}{4\pi} \right)^{1/2} P_{\ell}(\cos \theta(s)),$$

in terms of Legendre polynomials  $P_{\ell}$ , and satisfy the equations

$$\Delta_{\mathbb{S}^2} Y_{\ell 0} = -\ell(\ell + 1)Y_{\ell 0}, \quad \mathcal{L}_{\sigma} Y_{\ell 0} = 0.$$

Notice that, modulo a global constant that we absorb in the functions  $y_{\ell}$ , we have no other freedom in the choice of the angular part of the modes  $\phi_{\ell}(t, s) = y_{\ell}(t)Y_{\ell 0}(s)$ . The coefficients  $a_{\ell}$  must be subject, of course, to appropriate fall-off conditions in order to guarantee the pointwise convergence of the previous series. We also need a suitable norm in this space to talk about convergence. Though it is possible to detail at this stage the necessary structures and conditions we will not do so because the final construction of the quantum Hilbert space that we carry out is insensitive to these choices. Notice that (2.6), where only the  $Y_{\ell 0}$  harmonics appear, already takes into account the extra symmetry in the  $\sigma^a$  direction.

The massless Klein-Gordon equation leads now to the following equation for the complex functions  $y_{\ell}(t)$

$$\ddot{y}_{\ell} + (\cot t)\dot{y}_{\ell} + \ell(\ell + 1)y_{\ell} = 0. \quad (2.7)$$

We will always assume that, for each  $\ell$ , the real and imaginary parts of  $y_{\ell}$ ,  $u_{\ell}$  and  $v_{\ell}$  respectively, are two real linearly independent solutions of (2.7). We will not make at this point any specific choice for these functions but we will fix their normalization in the following way. Let us substitute first (2.6) in the symplectic structure  $\Omega$ . We find that

$$\Omega(\phi_1, \phi_2) = \sin t \sum_{\ell=0}^{\infty} (\bar{a}_{1\ell} a_{2\ell} - \bar{a}_{2\ell} a_{1\ell}) (y_{\ell}(t)\dot{\bar{y}}_{\ell}(t) - \dot{y}_{\ell}(t)\bar{y}_{\ell}(t)).$$

The previous expression can be simplified by first expanding  $y_{\ell}(t) = u_{\ell}(t) + iv_{\ell}(t)$  and writing

$$y_{\ell}(t)\dot{\bar{y}}_{\ell}(t) - \dot{y}_{\ell}(t)\bar{y}_{\ell}(t) = 2i \det \begin{pmatrix} \dot{u}_{\ell}(t) & u_{\ell}(t) \\ \dot{v}_{\ell}(t) & v_{\ell}(t) \end{pmatrix} =: 2iW(t; u_{\ell}, v_{\ell}).$$

As a consequence of the fact that  $y_{\ell}$  satisfies the differential equation (2.7) the Wronskian  $W$  satisfies

$$\dot{W} + (\cot t)W = 0 \Rightarrow W(t; u_{\ell}, v_{\ell}) = \frac{c_{\ell}}{\sin t}, \quad c_{\ell} \in \mathbb{R},$$

and hence the symplectic structure has the simple expression

$$\Omega(\phi_1, \phi_2) = 2i \sum_{\ell=0}^{\infty} c_{\ell} (\bar{a}_{1\ell} a_{2\ell} - \bar{a}_{2\ell} a_{1\ell}). \quad (2.8)$$

Notice that the time independence of the symplectic structure is explicit now. In the following we will choose the pair of functions  $(u_{\ell}, v_{\ell})$  normalized in such a way that  $c_{\ell} = 1/2$ ,  $\forall \ell$ , i.e.

$$W(t; u_{\ell}, v_{\ell}) = \frac{1}{2 \sin t}, \quad \forall (u_{\ell}, v_{\ell}), \quad \ell \in \mathbb{N} \cup \{0\}. \quad (2.9)$$

This condition is imposed in order to ensure that the modes  $\{\phi_{\ell}\}_{\ell=0}^{\infty}$  define an orthogonal basis of the one-particle Hilbert space on which we will construct the Fock space for the

quantum theory.

With the aim of characterizing the freedom in the election of the functions  $y_\ell$ , let us fix a specific family

$$\{y_{0\ell} = u_{0\ell} + iv_{0\ell} \mid \ell \in \mathbb{N} \cup \{0\}\} \quad (2.10)$$

satisfying the normalization condition<sup>7</sup> (2.9)

$$y_{0\ell} \dot{\bar{y}}_{0\ell} - \dot{y}_{0\ell} \bar{y}_{0\ell} = \frac{i}{\sin t}. \quad (2.11)$$

For any other normalized election of a family of linearly independent functions  $\{y_\ell = u_\ell + iv_\ell\}_{\ell=0}^\infty$  we can write (in terms of the  $u_{0\ell}$  and  $v_{0\ell}$ )

$$y_\ell(t) = u_\ell(t) + iv_\ell(t) = \alpha_\ell u_{0\ell}(t) + \beta_\ell v_{0\ell}(t) + i[\gamma_\ell u_{0\ell}(t) + \delta_\ell v_{0\ell}(t)]. \quad (2.12)$$

The normalization that we are choosing (2.11) gives the following condition for the real coefficients  $\alpha_\ell$ ,  $\beta_\ell$ ,  $\gamma_\ell$ , and  $\delta_\ell$

$$\alpha_\ell \delta_\ell - \beta_\ell \gamma_\ell = 1, \quad \ell \in \mathbb{N} \cup \{0\} \quad (2.13)$$

i.e.

$$\begin{pmatrix} \alpha_\ell & \beta_\ell \\ \gamma_\ell & \delta_\ell \end{pmatrix} \in SL(2; \mathbb{R}), \quad \ell \in \mathbb{N} \cup \{0\}.$$

As a set,  $SL(2, \mathbb{R})$  is in one-to-one correspondence with  $\mathbb{S}^1 \times \mathbb{R}^2$  and thus its elements can be factorized as

$$SL(2, \mathbb{R}) \ni \begin{pmatrix} \alpha_\ell & \beta_\ell \\ \gamma_\ell & \delta_\ell \end{pmatrix} = \begin{pmatrix} \cos \theta_\ell & -\sin \theta_\ell \\ \sin \theta_\ell & \cos \theta_\ell \end{pmatrix} \begin{pmatrix} \rho_\ell & \nu_\ell \\ 0 & \rho_\ell^{-1} \end{pmatrix} \quad (2.14)$$

for a unique choice of  $\rho_\ell > 0$ ,  $\nu_\ell \in \mathbb{R}$ ,  $\theta_\ell \in [0, 2\pi)$ . We will show in section III that the rotation part defined by the angle  $\theta_\ell$  plays a trivial role in the quantization of the model. As a consequence of this we will concentrate on the other factor involving  $\rho_\ell$  and  $\nu_\ell$ ,

$$y_\ell(t) = \rho_\ell u_{0\ell}(t) + (\nu_\ell + i\rho_\ell^{-1})v_{0\ell}(t), \quad (2.15)$$

and choose

$$u_{0\ell}(t) = \frac{1}{\sqrt{2}}P_\ell(\cos t), \quad v_{0\ell}(t) = \frac{1}{\sqrt{2}}Q_\ell(\cos t), \quad \ell \in \mathbb{N} \cup \{0\} \quad (2.16)$$

with  $P_\ell$  and  $Q_\ell$  denoting the first and second class Legendre functions, respectively. As we will see in section III B, the different choices of  $\{(\rho_\ell, \nu_\ell) \mid \ell \in \mathbb{N} \cup \{0\}\}$  in (2.15) will parameterize convenient complex structures that will allow us to construct the Fock representations for the quantum counterpart of the system.

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<sup>7</sup> Though it is possible to choose a normalization with the opposite sign ( $i \mapsto -i$ ), it is irrelevant as far as the unitarity issues discussed here are concerned, and amounts to interchanging negative and positive frequencies.

## B. Classical dynamics

Let us consider now the classical time evolution of the system. Given two values of the time parameter  $0 < t_0 \leq t_1 < \pi$ , the evolution from  $t_0$  to  $t_1$  can be viewed as a symplectomorphism  $\mathcal{T}_{(t_0, t_1)} : \Gamma \rightarrow \Gamma$  in the covariant phase space. It is possible to write  $\mathcal{T}_{(t_0, t_1)} = \mathfrak{J}_{t_0} \circ \mathfrak{J}_{t_1}^{-1}$  in terms of the maps  $\mathfrak{J}_t$  that, given a value of  $t$ , identify the space of Cauchy data  $\Upsilon$  with the covariant phase space  $\Gamma$ . This application (i) takes a solution of  $\mathcal{S}$ , (ii) finds the Cauchy data that this solution induces on  $\iota_{t_1}(\mathbb{S}^2)$  by virtue of the variational principle, (iii) imposes them as initial data on  $\iota_{t_0}(\mathbb{S}^2)$ , and (iv) finally finds the corresponding solution of  $\mathcal{S}$ . Explicitly, given

$$\phi(t, s) = \sum_{\ell=0}^{\infty} \left( a_{\ell} y_{\ell}(t) Y_{\ell 0}(s) + \overline{a_{\ell} y_{\ell}(t) Y_{\ell 0}(s)} \right) \in \Gamma, \quad (2.17)$$

the map

$$\mathfrak{J}_{t_1}^{-1} : \Gamma \rightarrow \Upsilon, \quad \phi \mapsto (Q, P) = \mathfrak{J}_{t_1}^{-1}(\phi) \quad (2.18)$$

is defined by

$$\begin{aligned} Q(s) &:= \phi(t_1, s) = \sum_{\ell=0}^{\infty} \left( a_{\ell} y_{\ell}(t_1) Y_{\ell 0}(s) + \overline{a_{\ell} y_{\ell}(t_1) Y_{\ell 0}(s)} \right), \\ P(s) &:= \sin t_1 \dot{\phi}(t_1, s) = \sin t_1 \sum_{\ell=0}^{\infty} \left( a_{\ell} \dot{y}_{\ell}(t_1) Y_{\ell 0}(s) + \overline{a_{\ell} \dot{y}_{\ell}(t_1) Y_{\ell 0}(s)} \right). \end{aligned} \quad (2.19)$$

On the other hand

$$\mathfrak{J}_{t_0} : \Upsilon \rightarrow \Gamma, \quad (Q, P) \mapsto \phi = \mathfrak{J}_{t_0}(Q, P) \quad (2.20)$$

is defined, in terms of the Fourier coefficients  $a_{\ell}$  of  $\phi$  (2.17), by

$$a_{\ell}(t_0) = -i \sin t_0 \dot{y}_{\ell}(t_0) \int_{\mathbb{S}^2} |\gamma|^{1/2} Y_{\ell 0} Q + i \bar{y}_{\ell}(t_0) \int_{\mathbb{S}^2} |\gamma|^{1/2} Y_{\ell 0} P. \quad (2.21)$$

By using (2.19) and (2.21) we finally get

$$\begin{aligned} (\mathcal{T}_{(t_0, t_1)} \phi)(t, s) &:= (\mathfrak{J}_{t_0} \circ \mathfrak{J}_{t_1}^{-1} \phi)(t, s) \\ &= \sum_{\ell=0}^{\infty} \left( \mathbf{a}_{\ell}(t_0, t_1) y_{\ell}(t) Y_{\ell 0}(s) + \overline{\mathbf{a}_{\ell}(t_0, t_1) y_{\ell}(t) Y_{\ell 0}(s)} \right) \end{aligned} \quad (2.22)$$

where

$$\begin{aligned} \mathbf{a}_{\ell}(t_0, t_1) &:= -i [\sin t_0 y_{\ell}(t_1) \dot{y}_{\ell}(t_0) - \sin t_1 \bar{y}_{\ell}(t_0) \dot{y}_{\ell}(t_1)] a_{\ell} \\ &\quad - i [\sin t_0 \bar{y}_{\ell}(t_1) \dot{y}_{\ell}(t_0) - \sin t_1 y_{\ell}(t_0) \dot{y}_{\ell}(t_1)] \bar{a}_{\ell}. \end{aligned} \quad (2.23)$$

In the next sections we will try to find out if this classical evolution can be unitarily implemented in a Fock quantization of the system.

### III. FOCK QUANTIZATION

In the passage to the quantum theory we have to introduce a Hilbert space for our system that we will write as  $\bigotimes_{i=0}^N \mathcal{F}_i$ . The Hilbert spaces  $\mathcal{F}_i$  will be used to describe the gravitational modes ( $i = 0$ ) and the massless scalar fields ( $i \in \mathbb{N}$ ). These will be taken to be symmetric Fock spaces built from appropriate one particle Hilbert spaces. As they are all isomorphic, and all the massless scalars satisfy the same equation, the same construction will be valid for all of them so we will omit the  $i$  index in the following. Here we will follow the quantization steps discussed in section 2.3 of reference [19] in order to define a suitable separable physical Hilbert space for the quantum theory, as well as irreducible representations for the canonical commutation relations. As expected for scalar fields in non-stationary curved background space-times, the Fock representation obtained in this way is highly non-unique.

In order to define the one-particle Hilbert space used to build the Fock space  $\mathcal{F}$ , let  $\mathcal{S}_{\mathbb{C}} := \mathbb{C} \otimes \mathcal{S}$  denote the  $\mathbb{C}$ -vector space obtained by the complexification of the solution space  $\mathcal{S}$  introduced above (2.3). The elements of  $\mathcal{S}_{\mathbb{C}}$  are ordered pairs of objects  $(\phi_1, \phi_2) \in \mathcal{S} \times \mathcal{S}$  that we will write in the form<sup>8</sup>  $\Phi := \phi_1 + i\phi_2$  with the natural definition for their sum. Multiplication by complex scalars  $\mathbb{C} \ni \lambda = \lambda_1 + i\lambda_2$ ,  $\lambda_1, \lambda_2 \in \mathbb{R}$ , is defined as

$$\lambda\Phi := (\lambda_1\phi_1 - \lambda_2\phi_2) + i(\lambda_2\phi_1 + \lambda_1\phi_2).$$

We also introduce the conjugation  $\bar{\cdot} : \mathcal{S}_{\mathbb{C}} \rightarrow \mathcal{S}_{\mathbb{C}} : (\phi_1 + i\phi_2) \mapsto (\phi_1 - i\phi_2)$ . Vectors in  $\mathcal{S}_{\mathbb{C}}$  can be expanded with the help of the basis  $\{\phi_\ell := y_\ell Y_{\ell 0}, \bar{\phi}_\ell := \overline{y_\ell Y_{\ell 0}}\}_{\ell=0}^\infty$  introduced above as

$$\Phi = \sum_{\ell=0}^{\infty} \left( a_\ell y_\ell Y_{\ell 0} + b_\ell \overline{y_\ell Y_{\ell 0}} \right)$$

with  $a_\ell, b_\ell \in \mathbb{C}$ . The symplectic structure (2.4) defined on  $\mathcal{S}$  can be extended in a linear way to  $\mathcal{S}_{\mathbb{C}}$  as

$$\Omega_{\mathbb{C}}(\Phi_1, \Phi_2) := i \sum_{\ell=0}^{\infty} \left( b_{1\ell} a_{2\ell} - b_{2\ell} a_{1\ell} \right).$$

For each pair  $\Phi_1, \Phi_2 \in \mathcal{S}_{\mathbb{C}}$  the mapping

$$\langle \cdot | \cdot \rangle : \mathcal{S}_{\mathbb{C}} \times \mathcal{S}_{\mathbb{C}} \rightarrow \mathbb{C}, \quad (\Phi_1, \Phi_2) \mapsto \langle \Phi_1 | \Phi_2 \rangle := -i\Omega_{\mathbb{C}}(\bar{\Phi}_1, \Phi_2) \quad (3.1)$$

is antilinear in the first argument and linear in the second. It is not an inner product because it is not positive-definite. There are, however, linear subspaces of  $\mathcal{S}_{\mathbb{C}}$  where  $\langle \cdot | \cdot \rangle$  is positive definite (and, hence, defines an inner product). Let us consider, in particular, the Lagrangian subspace

$$\mathcal{P} := \left\{ \Phi \in \mathcal{S}_{\mathbb{C}} \mid \Phi = \sum_{\ell=0}^{\infty} a_\ell \phi_\ell \right\}. \quad (3.2)$$

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<sup>8</sup> Here  $i \in \mathbb{C}$  denotes the imaginary unit.



Here the restriction  $\langle \cdot | \cdot \rangle_{\mathcal{P}}$  defines an inner product given by

$$\langle \Phi_1 | \Phi_2 \rangle = \sum_{\ell=0}^{\infty} \bar{a}_{1\ell} a_{2\ell}, \quad \Phi_1, \Phi_2 \in \mathcal{P}. \quad (3.3)$$

The one particle Hilbert space  $\mathcal{H}_{\mathcal{P}}$  is then the Cauchy completion of  $(\mathcal{P}, \langle \cdot | \cdot \rangle_{\mathcal{P}})$  w.r.t. the norm defined by the inner product. Notice that the set  $\{\phi_{\ell} = y_{\ell} Y_{\ell 0} \mid \ell \in \mathbb{N} \cup \{0\}\}$  becomes an orthonormal basis of  $\mathcal{H}_{\mathcal{P}}$  satisfying  $\langle \phi_{\ell_1} | \phi_{\ell_2} \rangle = \delta(\ell_1, \ell_2)$ . Finally, the quantum Hilbert space<sup>9</sup> is given by the symmetric Fock space

$$\mathcal{F}_s(\mathcal{H}_{\mathcal{P}}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_{\mathcal{P}}^{\otimes n},$$

where  $\mathcal{H}_{\mathcal{P}}^{\otimes 0} := \mathbb{C}$ , and  $\mathcal{H}_{\mathcal{P}}^{\otimes n}$  denotes the subspace of  $\mathcal{H}_{\mathcal{P}}^{\otimes n} = \bigotimes_{k=1}^n \mathcal{H}_{\mathcal{P}}$  spanned by symmetric tensor products of  $n$  vectors in  $\mathcal{H}_{\mathcal{P}}$  (these are referred to as  $n$ -particle subspaces). Associated to the modes  $\phi_{\ell} \in \mathcal{H}_{\mathcal{P}}$  we have the corresponding annihilation  $\hat{a}_{\ell}$  and creation operators  $\hat{a}_{\ell}^{\dagger}$ , with non-vanishing commutation relations given by  $[\hat{a}_{\ell_1}, \hat{a}_{\ell_2}^{\dagger}] = \delta(\ell_1, \ell_2)$ . As usual, we will denote as  $|0\rangle$  the Fock vacuum  $1 \oplus 0 \oplus 0 \oplus \dots \in \mathcal{F}_s(\mathcal{H}_{\mathcal{P}})$  whose only nonzero component is  $1 \in \mathbb{C}$  and we will use a subindex  $\mathcal{P}$  whenever we have to emphasize the dependence of these objects on the subspace  $\mathcal{P}$ . The Fock vacuum  $|0\rangle$  is in the domain of all finite products of creation and annihilation operators and the vectors

$$|{}^1 n_{\ell_1} {}^2 n_{\ell_2} \dots {}^k n_{\ell_k}\rangle := \frac{1}{\sqrt{{}^1 n! {}^2 n! \dots {}^k n!}} (\hat{a}_{\ell_1}^{\dagger})^{{}^1 n} (\hat{a}_{\ell_2}^{\dagger})^{{}^2 n} \dots (\hat{a}_{\ell_k}^{\dagger})^{{}^k n} |0\rangle \in \mathcal{F}_s(\mathcal{H}_{\mathcal{P}}),$$

where  $k \in \mathbb{N} \cup \{0\}$ ,  $({}^1 n, {}^2 n, \dots, {}^k n) \in \mathbb{N}^k$ , and  $\ell_i \neq \ell_j$  for  $i \neq j$ , provide a basis of  $\mathcal{F}_s(\mathcal{H}_{\mathcal{P}})$ . The basis vectors are normalized according to

$$\langle {}^1 n_{\ell_1} \dots {}^k n_{\ell_k} | {}^1 m_{\ell'_1} \dots {}^r m_{\ell'_r} \rangle = \delta(k, r) \sum_{\pi \in S_k} \delta({}^1 n, \pi^{(1)} m) \dots \delta({}^k n, \pi^{(k)} m) \delta(\ell_1, \ell'_{\pi(1)}) \dots \delta(\ell_k, \ell'_{\pi(k)}),$$

where  $S_k$  denotes the set of permutations  $\pi$  of the  $k$  symbols  $\{1, 2, \dots, k\}$ . Also, they satisfy

$$\hat{a}_{\ell}^{\dagger} |n_{\ell}\rangle = \sqrt{n+1} |(n+1)_{\ell}\rangle, \quad \hat{a}_{\ell} |n_{\ell}\rangle = \sqrt{n} |(n-1)_{\ell}\rangle.$$

Notice that, using the notation introduced above, the modes  $\phi_{\ell}$  of the one particle Hilbert space  $\mathcal{H}_{\mathcal{P}}$  can now be considered as one-particle states that we will denote as  $|1_{\ell}\rangle := \hat{a}_{\ell}^{\dagger} |0\rangle \in \mathcal{F}_s(\mathcal{H}_{\mathcal{P}})$ .

### A. Complex structures

The previous construction for the one-particle Hilbert space is based on a non-unique choice (3.2) for the subspace  $\mathcal{P}$  of what are usually called “positive frequency” solutions to

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<sup>9</sup> At variance with the  $\mathbb{T}^3$  case where some constraints must be taken into account we do not have any in this case.

the field equations. Since we are dealing with non-stationary space-times, it is not possible to select a natural subspace  $\mathcal{P}$  by invoking a time translation symmetry. Furthermore, the deparameterization procedure does not provide extra constraints [17] that would generate residual symmetries useful to define a preferred choice of  $\mathcal{P}$ . This fact manifests itself as an ambiguity in the formulation of the quantum theory, because different choices of  $\mathcal{P}$  generally yield unitarily inequivalent Fock representations [19]. We will show here that every possible choice of the subspace  $\mathcal{P}$  defined in (3.2) is in correspondence with a  $SO(3)$ -invariant complex structure on the solution space  $\mathcal{S}$ , postponing to section IV B a discussion of the uniqueness of the representation.

An equivalent way to deal with the splitting  $\mathcal{S}_{\mathbb{C}} = \mathcal{P} \oplus \bar{\mathcal{P}}$  is to introduce a complex structure  $J : \mathcal{S}_{\mathbb{C}} \rightarrow \mathcal{S}_{\mathbb{C}}$ , and define  $\mathcal{P}$  and  $\bar{\mathcal{P}}$  as the eigenspaces associated to the eigenvalues  $+i$  and  $-i$ , respectively. The complex structure must satisfy the following conditions

- J1)  $J$  is a  $\mathbb{C}$ -linear map  $J : \mathcal{S}_{\mathbb{C}} \rightarrow \mathcal{S}_{\mathbb{C}}$  satisfying  $J^2 = -\text{Id}_{\mathcal{S}_{\mathbb{C}}}$ .
- J2)  $J$  induces a  $\mathbb{R}$ -linear map  $\mathcal{S} \rightarrow \mathcal{S}$  i.e.  $\overline{J\Phi} = J\bar{\Phi}$  for all  $\Phi \in \mathcal{S}_{\mathbb{C}}$ .
- J3) The sesquilinear form (3.1) restricted to the subspace corresponding to the  $i$  eigenvalue of  $J$  (that we denote as  $\mathcal{P}$ ) defines an inner product.

In practice this complex structure is defined once a choice of modes like the one introduced above is given. For example, if we consider the family  $\{y_{0\ell}\}_{\ell=0}^{\infty}$  given by (2.10) and (2.16), the set of functions  $\{\phi_{0\ell} = y_{0\ell}Y_{\ell}\}_{\ell=0}^{\infty}$  allows us to define a complex structure by

$$J_0\phi_{0\ell} := i\phi_{0\ell}, \quad J_0\bar{\phi}_{0\ell} := -i\bar{\phi}_{0\ell}.$$

We will denote the vector spaces generated by  $\phi_{0\ell}$  and  $\bar{\phi}_{0\ell}$  as  $\mathcal{P}_0$  and  $\bar{\mathcal{P}}_0$  respectively. In principle a different choice for  $\{y_{\ell}\}_{\ell=0}^{\infty}$  would give rise to a different complex structure. However this is not always the case. For example, if we obtain  $y_{\ell}$  from  $y_{0\ell}$  by the rotation appearing in the decomposition (2.14) of the  $SL(2, \mathbb{R})$  matrices discussed above

$$y_{\ell} = u_{\ell} + iv_{\ell} = \cos\theta_{\ell}u_{0\ell} - \sin\theta_{\ell}v_{0\ell} + i(\sin\theta_{\ell}u_{0\ell} + \cos\theta_{\ell}v_{0\ell}) = e^{i\theta_{\ell}}y_{0\ell}$$

the set  $\{\phi_{\ell} = y_{\ell}Y_{\ell}\}_{\ell=0}^{\infty}$  defines a complex structure  $J$  through

$$J\phi_{\ell} := i\phi_{\ell}, \quad J\bar{\phi}_{\ell} := -i\bar{\phi}_{\ell}.$$

Now it is straightforward to see that  $J\phi_{\ell} = i\phi_{\ell} \Leftrightarrow J e^{i\theta_{\ell}}\phi_{0\ell} = i e^{i\theta_{\ell}}\phi_{0\ell}$  and  $\mathbb{C}$ -linearity implies  $J\phi_{0\ell} = i\phi_{0\ell}$  i.e.  $J = J_0$ .

Given the decomposition  $\mathcal{S}_{\mathbb{C}} = \mathcal{P}_0 \oplus \bar{\mathcal{P}}_0$  there are two antilinear maps that connect the spaces  $\mathcal{P}_0$  and  $\bar{\mathcal{P}}_0$  that we denote (in a slight notational abuse) with the same symbol  $\bar{\cdot} : \mathcal{P}_0 \rightarrow \bar{\mathcal{P}}_0 : \psi_1 \mapsto \bar{\psi}_1$  and  $\bar{\cdot} : \bar{\mathcal{P}}_0 \rightarrow \mathcal{P}_0 : \psi_2 \mapsto \bar{\psi}_2$ . Each one of these maps is the inverse of the other and their composition is the identity for every element of  $\mathcal{P}_0$  or  $\bar{\mathcal{P}}_0$  (i.e.  $\bar{\bar{\psi}} = \psi$ ). With their help we can write the conjugation  $\bar{\cdot} : \mathcal{S}_{\mathbb{C}} \rightarrow \mathcal{S}_{\mathbb{C}}$  according to

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \mapsto \bar{\Psi} := \begin{pmatrix} \bar{\psi}_2 \\ \bar{\psi}_1 \end{pmatrix}$$

with  $\psi_1 \in \mathcal{P}_0$  and  $\psi_2 \in \bar{\mathcal{P}}_0$ . The elements in the original (real) solution space  $\mathcal{S}$  can be easily

characterized by using the previous conjugation as those of the form

$$\Phi = \begin{pmatrix} \phi \\ \bar{\phi} \end{pmatrix}$$

or, alternatively, as the real linear subspace of  $\mathcal{S}_{\mathbb{C}}$  given by  $\mathcal{S} = \{\Phi \in \mathcal{S}_{\mathbb{C}} \mid \Phi = \bar{\Phi}\}$ .

Let us characterize now the complex structures in  $\mathcal{S}_{\mathbb{C}}$  with the help of the fixed decomposition introduced above  $\mathcal{S}_{\mathbb{C}} = \mathcal{P}_0 \oplus \bar{\mathcal{P}}_0$ . In particular, every  $\mathbb{C}$ -linear map  $J : \mathcal{S}_{\mathbb{C}} \rightarrow \mathcal{S}_{\mathbb{C}}$  can be written in the form

$$J = \begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix},$$

where the maps  $J_{ab} : \mathcal{P}_b \rightarrow \mathcal{P}_a$  are  $\mathbb{C}$ -linear for  $a, b \in \{1, 2\}$ , and we have introduced the convenient notation  $\mathcal{P}_1 := \mathcal{P}_0$  and  $\mathcal{P}_2 := \bar{\mathcal{P}}_0$ . The necessary and sufficient condition to guarantee that the restriction of  $J$  to  $\mathcal{S}$  is  $\mathbb{R}$ -linear is  $J\Phi = \overline{J\Phi}$  for every  $\Phi \in \mathcal{S}$ , or equivalently

$$J_{11}\phi = \overline{J_{22}\bar{\phi}}, \quad J_{21}\phi = \overline{J_{12}\bar{\phi}},$$

i.e.

$$J_{22} = \bar{J}_{11}, \quad J_{12} = \bar{J}_{21}, \quad (3.4)$$

where we have used the notation  $\bar{A}\phi := \overline{A\bar{\phi}}$  to denote the  $\mathbb{C}$ -linear map  $\bar{A} : \mathcal{P}_b \rightarrow \mathcal{P}_a$  ( $a \neq b$ ) obtained from the  $\mathbb{C}$ -linear map  $A : \mathcal{P}_a \rightarrow \mathcal{P}_b$ . Finally the condition  $J^2 = -\text{Id}_{\mathcal{S}_{\mathbb{C}}}$  requires that

$$J_{11}^2 + \bar{J}_{21}J_{21} = -\text{Id}_1, \quad J_{21}J_{11} + \bar{J}_{11}J_{21} = 0.$$

We will see in the next subsection how the symmetries of the problem help us fix the form of the  $J_{ab}$ .

## B. Invariant complex structures

Here we want to characterize those complex structures in the solution space  $\mathcal{S}^{KG}$  of the field equation  $\mathring{g}^{ab}\mathring{\nabla}_a\mathring{\nabla}_b\phi = 0$ , invariant under the symmetries of  $\mathbb{S}^2$ —the spatial manifold in our  $(2+1)$ -dimensional description—*without imposing the condition*  $\mathcal{L}_\sigma\phi = 0$ . As we will show, once this is done it is straightforward to restrict them to the solution space  $\mathcal{S}$ . To this end let us consider the complexified solution space  $\mathcal{S}_{\mathbb{C}}^{KG} = \mathcal{P}_0^{KG} \oplus \bar{\mathcal{P}}_0^{KG}$  where

$$\begin{aligned} \mathcal{P}_1^{KG} &:= \mathcal{P}_0^{KG} = \text{span}\{y_{0\ell}Y_{\ell m} \mid \ell \in \mathbb{N} \cup \{0\}, m \in \{-\ell, \dots, \ell\}\}, \\ \mathcal{P}_2^{KG} &:= \bar{\mathcal{P}}_0^{KG} = \text{span}\{\bar{y}_{0\ell}Y_{\ell m} \mid \ell \in \mathbb{N} \cup \{0\}, m \in \{-\ell, \dots, \ell\}\}. \end{aligned}$$

Here  $Y_{\ell m}$  are the usual spherical harmonics on  $\mathbb{S}^2$ .

The elements  $\phi_a \in \mathcal{P}_a^{KG}$ ,  $a = 1, 2$ , are complex functions  $\phi_a(t, s)$  defined on  $(0, \pi) \times \mathbb{S}^2$ . There is a natural representation  $D_a$  of  $SO(3)$  in  $\mathcal{P}_a^{KG}$  defined by  $(D_a(g)\phi)(t, s) = \phi(t, g^{-1} \cdot s)$  where  $g^{-1} \cdot s$  denotes the action of the rotation  $g^{-1} \in SO(3)$  on the point  $s \in \mathbb{S}^2$ . Then the natural representation of  $SO(3)$  in  $\mathcal{S}_{\mathbb{C}}^{KG} = \mathcal{P}_1^{KG} \oplus \mathcal{P}_2^{KG}$  can be written in matrix form as

$$D(g) = \begin{pmatrix} D_1(g) & 0 \\ 0 & D_2(g) \end{pmatrix}, \quad g \in SO(3),$$

in terms of the representations  $(D_a, \mathcal{P}_a^{KG})$ . The invariance of a  $\mathbb{C}$ -linear map  $J$  under the action of the group  $SO(3)$  implies

$$D(g)J = JD(g) \Leftrightarrow \begin{pmatrix} J_{11}D_1(g) & J_{12}D_2(g) \\ J_{21}D_1(g) & J_{22}D_2(g) \end{pmatrix} = \begin{pmatrix} D_1(g)J_{11} & D_1(g)J_{12} \\ D_2(g)J_{21} & D_2(g)J_{22} \end{pmatrix}, \quad \forall g \in SO(3).$$

It is convenient now to expand the vector spaces  $\mathcal{P}_a^{KG}$  as

$$\mathcal{P}_a^{KG} = \bigoplus_{\ell=0}^{\infty} \mathcal{P}_a^{\ell}, \quad a = 1, 2, \quad (3.5)$$

with

$$\begin{aligned} \mathcal{P}_1^{\ell} &:= \text{span}\{y_{0\ell}\} \otimes \text{span}\{Y_{\ell m} \mid m \in \{-\ell, \dots, \ell\}\}, \\ \mathcal{P}_2^{\ell} &:= \text{span}\{\bar{y}_{0\ell}\} \otimes \text{span}\{Y_{\ell m} \mid m \in \{-\ell, \dots, \ell\}\}. \end{aligned}$$

This is useful because the operators  $D_a(g)$  can be written as  $D_a = \bigoplus_{\ell=0}^{\infty} D_a^{\ell}$ , where each of the  $(\mathcal{P}_a^{\ell}, D_a^{\ell})$  are irreducible representations.

Denoting as  $\Pi_a^{\ell}$  the projectors on the linear spaces  $\mathcal{P}_a^{\ell}$  we can write the linear mappings  $J_{ab}$  as

$$J_{ab}^{\ell_1 \ell_2} := \Pi_a^{\ell_1} J_{ab} \Pi_b^{\ell_2} : \mathcal{P}_b^{\ell_2} \rightarrow \mathcal{P}_a^{\ell_1}.$$

We use now Schur's lemma<sup>10</sup> that directly implies that  $J_{ab}^{\ell_1 \ell_2} = 0$  whenever  $\ell_1 \neq \ell_2$ ,  $J_{aa}^{\ell\ell} = j_{aa}^{\ell} I_{aa}^{\ell}$  where  $j_{aa}^{\ell} \in \mathbb{C}$ , ( $I_{aa}^{\ell}$  denotes the identity on  $\mathcal{P}_a^{\ell}$ ) and  $j_{22}^{\ell} = \bar{j}_{11}^{\ell}$  as a consequence of (3.4). Also

$$J_{12}^{\ell\ell}(\bar{y}_{0\ell} \otimes v) = j_{12}^{\ell} y_{0\ell} \otimes v, \quad J_{21}^{\ell\ell}(y_{0\ell} \otimes v) = \bar{j}_{21}^{\ell} \bar{y}_{0\ell} \otimes v, \quad j_{12}^{\ell}, \bar{j}_{21}^{\ell} \in \mathbb{C}$$

with  $j_{12}^{\ell} = \bar{j}_{21}^{\ell}$  again as a consequence of (3.4). In conclusion the general form of the mapping  $J$  is given by

$$J = \bigoplus_{\ell=0}^{\infty} \begin{pmatrix} j_{11}^{\ell} I_{11}^{\ell} & j_{12}^{\ell} I_{12}^{\ell} \\ \bar{j}_{12}^{\ell} I_{21}^{\ell} & \bar{j}_{11}^{\ell} I_{22}^{\ell} \end{pmatrix},$$

where  $I_{aa}^{\ell}$  denotes the identity operator in  $\mathcal{P}_a^{\ell}$  and the linear operators  $I_{ab}^{\ell} : \mathcal{P}_b^{\ell} \rightarrow \mathcal{P}_a^{\ell}$  act according to  $I_{12}^{\ell}(\bar{y}_{0\ell} \otimes v) = y_{0\ell} \otimes v$  and  $I_{21}^{\ell}(y_{0\ell} \otimes v) = \bar{y}_{0\ell} \otimes v$ .

The condition  $J^2 = -\text{Id}_{\mathcal{S}^{KG}}$  defining  $J$  as a complex structure gives finally the following restriction on  $j_{11}^{\ell}$  and  $j_{12}^{\ell}$

$$|j_{11}^{\ell}|^2 - |j_{12}^{\ell}|^2 = 1, \quad j_{11}^{\ell} \in i\mathbb{R} \setminus \{0\}, \quad j_{12}^{\ell} \in \mathbb{C}. \quad (3.6)$$

Several comments are in order now. First of all as we can see, on each subspace  $\mathcal{P}_1^{\ell} \oplus \mathcal{P}_2^{\ell}$  the complex structure is completely fixed by a pair of complex parameters  $(j_{11}^{\ell}, j_{12}^{\ell})$  subject to

<sup>10</sup> *Schur lemma:* Let  $D_1(g)$  and  $D_2(g)$  be two finite dimensional, irreducible representations of the group  $G$  in the complex finite-dimensional linear spaces  $V_1$  and  $V_2$ . Let us suppose that a linear operator  $L : V_1 \rightarrow V_2$  ‘commutes’ with these representations (i.e.  $D_2(g)L = LD_1(g)$ ,  $\forall g \in G$ ). Then either  $L$  is zero or it is invertible. In this last case both representations are equivalent and  $L$  is uniquely determined modulo a multiplicative constant.

the conditions (3.6); the remaining freedom is then parameterized by two real numbers. This is what we have found before by explicitly considering the solution space and the choice of the families of functions  $u_\ell$  and  $v_\ell$ . It is straightforward to check that the complex structures naturally defined by these families of functions are in fact  $SO(3)$  invariant. The previous argument then shows that they exhaust, in fact, all the possibilities. The choice  $j_{11}^\ell \in i\mathbb{R}_+$  is equivalent to the normalization for the Wronskian of  $u_\ell$  and  $v_\ell$  introduced above in equation (2.9) and guarantees that the condition J3 in section III A is satisfied. Changing the sign in the Wronskian corresponds to taking  $j_{11}^\ell \in i\mathbb{R}_-$ .

The previous considerations apply to solutions of the Klein-Gordon equation without imposing the additional axial symmetry. This can be trivially taken into account at this point by realizing that it suffices to restrict ourselves to the one-dimensional subspaces (for each value of  $\ell$ ) spanned by the spherical harmonics  $Y_{\ell 0}$ .

Finally we give here the formulas that relate the parameters  $\rho_\ell$  and  $\nu_\ell$  to the definition of the invariant complex structure discussed in this section. Once a fiducial basis  $\phi_{0\ell} = y_{0\ell}Y_{\ell 0}$  is chosen (2.10) any other complex structure defined by a different basis –satisfying the normalization condition (2.11)– can be written in terms of  $\phi_{0\ell}$ , by using (2.12) and (2.14), as

$$J \begin{pmatrix} \phi_{0\ell} \\ \bar{\phi}_{0\ell} \end{pmatrix} = \begin{pmatrix} j_{11}^\ell I_{11}^\ell & j_{12}^\ell I_{12}^\ell \\ j_{12}^\ell I_{21}^\ell & j_{11}^\ell I_{22}^\ell \end{pmatrix} \begin{pmatrix} \phi_{0\ell} \\ \bar{\phi}_{0\ell} \end{pmatrix}, \quad (3.7)$$

where

$$j_{11}^\ell = \frac{i}{2}(\alpha_\ell^2 + \beta_\ell^2 + \gamma_\ell^2 + \delta_\ell^2) = \frac{i}{2}(\nu_\ell^2 + \rho_\ell^{-2} + \rho_\ell^2), \quad (3.8)$$

$$j_{12}^\ell = -(\alpha_\ell\beta_\ell + \gamma_\ell\delta_\ell) + \frac{i}{2}(\beta_\ell^2 + \delta_\ell^2 - \alpha_\ell^2 - \gamma_\ell^2) = -\rho_\ell\nu_\ell + \frac{i}{2}(\nu_\ell^2 + \rho_\ell^{-2} - \rho_\ell^2). \quad (3.9)$$

Notice that, as expected, the complex structures defined by (3.8) and (3.9) do not depend on the parameters  $\theta_\ell \in [0, 2\pi)$  appearing in (2.14) but only on the pairs  $(\rho_\ell, \nu_\ell) \in (0, \infty) \times \mathbb{R}$ . Notice also that these last formulas relate the invariant complex structures described here with the ones obtained in section III A by studying the mode decomposition in the solution space.

#### IV. UNITARITY OF THE QUANTUM TIME EVOLUTION AND UNIQUENESS OF THE FOCK REPRESENTATION

We discuss in this section the unitarity of the quantum evolution for the classical system described in section II corresponding to the reduced phase space of the Gowdy models coupled to massless scalar fields with  $\mathbb{S}^1 \times \mathbb{S}^2$  and  $\mathbb{S}^3$  spatial topologies. We also study the uniqueness (after re-scaling of the field) of the Fock representation under the requirement, on the complex structures, of  $SO(3)$  invariance and unitarity of the dynamics.

It is well known [20] that not every linear symplectic transformation  $\mathcal{T}$  defined on the infinite dimensional symplectic linear space  $\Gamma$  can be unitarily implemented in a Fock quantization of the system. Let  $\mathcal{T} : \Gamma \rightarrow \Gamma$  be a *continuous* linear symplectic transformation.

Given any point

$$\phi = \sum_{\ell=0}^{\infty} \left( a_{\ell} y_{\ell} Y_{\ell 0} + \overline{a_{\ell} y_{\ell} Y_{\ell 0}} \right) \in \Gamma, \quad (4.1)$$

the action of  $\mathcal{T}$  can be written in the form

$$\mathcal{T}\phi = \sum_{\ell_1=0}^{\infty} \left( \mathbf{a}_{\ell_1}(a, \bar{a}) y_{\ell_1} Y_{\ell_1 0} + \overline{\mathbf{a}_{\ell_1}(a, \bar{a}) y_{\ell_1} Y_{\ell_1 0}} \right),$$

where the complex coefficients

$$\mathbf{a}_{\ell_1}(a, \bar{a}) = \sum_{\ell_2=0}^{\infty} \left( \alpha(\ell_1, \ell_2) a_{\ell_2} + \beta(\ell_1, \ell_2) \bar{a}_{\ell_2} \right)$$

must satisfy certain conditions to ensure the continuity.  $\mathcal{T}$  is implementable in the quantum theory as a unitary operator, i.e. there exists a unitary operator  $\hat{U}_{\mathcal{T}} : \mathcal{F}_s(\mathcal{H}_{\mathcal{P}}) \rightarrow \mathcal{F}_s(\mathcal{H}_{\mathcal{P}})$  such that

$$\hat{U}_{\mathcal{T}}^{-1} \hat{a}_{\ell_1} \hat{U}_{\mathcal{T}} = \sum_{\ell_2=0}^{\infty} \left( \alpha(\ell_1, \ell_2) \hat{a}_{\ell_2} + \beta(\ell_1, \ell_2) \hat{a}_{\ell_2}^{\dagger} \right),$$

if and only if  $J_{\mathcal{P}} - \mathcal{T}^{-1} \circ J_{\mathcal{P}} \circ \mathcal{T}$  is Hilbert-Schmidt (here  $J_{\mathcal{P}}$  is the complex structure associated to the  $\mathcal{P}$  subspace) [19, 20]. Equivalently this can be expressed as

$$\sum_{\ell_1=0}^{\infty} \sum_{\ell_2=0}^{\infty} |\beta(\ell_1, \ell_2)|^2 < \infty.$$

This condition for the unitary implementability of the symplectic transformation  $\mathcal{T}_{(t_0, t_1)}$ , that defines the time evolution on  $\Gamma$ , can be written from (2.22) and (2.23) as

$$\sum_{\ell=0}^{\infty} |\beta_{\ell}(t_0, t_1 | y_{\ell})|^2 = \sum_{\ell=0}^{\infty} |\sin t_0 y_{\ell}(t_1) \dot{y}_{\ell}(t_0) - \sin t_1 y_{\ell}(t_0) \dot{y}_{\ell}(t_1)|^2 < \infty, \quad \forall t_0, t_1 \in (0, \pi), \quad (4.2)$$

where  $\beta_{\ell}(t_0, t_1 | y_{\ell}) := \sin t_0 y_{\ell}(t_1) \dot{y}_{\ell}(t_0) - \sin t_1 y_{\ell}(t_0) \dot{y}_{\ell}(t_1)$ . At this point we have to study the convergence of the previous series. To this end let us consider the imaginary part of the coefficients  $\beta_{\ell}$ ; by using the expression (2.15) for  $y_{\ell}$  it is possible to identify the dependence of  $\text{Im}(\beta_{\ell}(t_0, t_1 | y_{\ell}))$  on the choice of complex structure –parameterized by  $(\rho_{\ell}, \nu_{\ell})$ . This is given by

$$\text{Im}(\beta_{\ell}(t_0, t_1 | y_{\ell})) = A_{\ell}(t_0, t_1) + 2\rho_{\ell}^{-1} \nu_{\ell} B_{\ell}(t_0, t_1), \quad (4.3)$$

where

$$\begin{aligned} A_{\ell}(t_0, t_1) &:= \sin t_0 [u_{0\ell}(t_1) \dot{v}_{0\ell}(t_0) + \dot{u}_{0\ell}(t_0) v_{0\ell}(t_1)] - \sin t_1 [u_{0\ell}(t_0) \dot{v}_{0\ell}(t_1) + v_{0\ell}(t_0) \dot{u}_{0\ell}(t_1)], \\ B_{\ell}(t_0, t_1) &:= \sin t_0 v_{0\ell}(t_1) \dot{v}_{0\ell}(t_0) - \sin t_1 v_{0\ell}(t_0) \dot{v}_{0\ell}(t_1). \end{aligned}$$

The explicit form of  $A_\ell$  and  $B_\ell$ , derived in a straightforward way from (2.16), is

$$\begin{aligned}
A_\ell(t_0, t_1) &= -\frac{\ell+1}{2} \left( P_{\ell+1}(\cos t_1) Q_\ell(\cos t_0) - P_{\ell+1}(\cos t_0) Q_\ell(\cos t_1) \right. \\
&\quad + P_\ell(\cos t_1) [(\cos t_0 - \cos t_1) Q_\ell(\cos t_0) - Q_{\ell+1}(\cos t_0)] \\
&\quad \left. + P_\ell(\cos t_0) [(\cos t_0 - \cos t_1) Q_\ell(\cos t_1) - Q_{\ell+1}(\cos t_1)] \right), \\
B_\ell(t_0, t_1) &= \frac{\ell+1}{2} \left( Q_\ell(\cos t_1) Q_{\ell+1}(\cos t_0) \right. \\
&\quad \left. - [(\cos t_0 - \cos t_1) Q_\ell(\cos t_1) + Q_{\ell+1}(\cos t_1)] Q_\ell(\cos t_0) \right).
\end{aligned} \tag{4.4}$$

By using the following asymptotic expansions for the first and second class Legendre functions ( $\varepsilon < t < \pi - \varepsilon$ ,  $\varepsilon > 0$ ) [21]

$$\begin{aligned}
P_\ell(\cos t) &= \frac{\Gamma(\ell+1)}{\Gamma(\ell+3/2)} \sqrt{\frac{2}{\pi \sin t}} \cos[(\ell+1/2)t - \pi/4] + O(\ell^{-1}), \\
Q_\ell(\cos t) &= \frac{\Gamma(\ell+1)}{\Gamma(\ell+3/2)} \sqrt{\frac{\pi}{2 \sin t}} \cos[(\ell+1/2)t + \pi/4] + O(\ell^{-1}),
\end{aligned} \tag{4.5}$$

we find that, for  $\ell \rightarrow \infty$ ,

$$\begin{aligned}
\text{Im}(\beta_\ell(t_0, t_1|y_\ell)) &\sim -\frac{1}{2} \frac{\sin t_0 - \sin t_1}{\sqrt{\sin t_0 \sin t_1}} \sin[(\ell+1/2)(t_0 + t_1)] \\
&\quad - \frac{\pi \nu_\ell \rho_\ell^{-1}}{2 \sqrt{\sin t_0 \sin t_1}} \left( \sin t_0 \cos[(\ell+1/2)t_1 + \pi/4] \sin[(\ell+1/2)t_0 + \pi/4] \right. \\
&\quad \left. - \sin t_1 \cos[(\ell+1/2)t_0 + \pi/4] \sin[(\ell+1/2)t_1 + \pi/4] \right).
\end{aligned}$$

The asymptotic behavior of  $\text{Im}(\beta_\ell(t_0, t_1|y_\ell))$  leads us to conclude that irrespective of the choice of  $(\rho_\ell, \nu_\ell)$  we have that  $\text{Im}(\beta_\ell(t_0, t_1|y_\ell))$  is not square summable and hence time evolution cannot be unitarily implemented for any choice of  $SO(3)$ -invariant complex structure.

### A. Conformal field redefinitions

We will show now that we can avoid this negative conclusion much in the same way as in the three-torus  $\mathbb{T}^3$  case, i.e. by introducing a redefinition of the fields in terms of which the model is formulated [13]. In our approach this redefinition is suggested by the functional form of the conformal factor  $\sin t$  appearing in the auxiliary metric  $\hat{g}_{ab}$  (2.1). In the following we will reintroduce the index  $i$  that labels the gravitational scalar ( $i = 0$ ) and the matter scalars ( $i = 1, \dots, N$ ) and consider the new fields

$$\xi_i := \sqrt{\sin t} \phi_i.$$

The field equations are now

$$-\ddot{\xi}_i + \Delta_{\mathbb{S}^2} \xi_i = \frac{1}{4} (1 + \csc^2 t) \xi_i, \quad \mathcal{L}_\sigma \xi_i = 0. \tag{4.6}$$

They can be interpreted as the equation for a scalar, axially symmetric field with a time-dependent mass term  $\frac{1}{4}(1 + \csc^2 t)$ , evolving in  $(0, \pi) \times \mathbb{S}^2$  with the *regular* –i.e. extensible to  $\mathbb{R} \times \mathbb{S}^2$ – background metric

$$\mathring{\eta}_{ab} = -(\mathrm{d}t)_a(\mathrm{d}t)_b + \gamma_{ab}.$$

Notice that the mass term is singular at  $t = 0$  and  $t = \pi$  but has the *correct sign* for all  $t \in (0, \pi)$ . This field redefinition can be incorporated in the model at the Lagrangian level by substituting  $\phi_i = \xi_i / \sqrt{\sin t}$  in the action (2.2) to get the corresponding variational problem in terms of the new fields

$$\begin{aligned} s(\xi_i) = & -\frac{1}{2} \sum_{i=0}^N \int_{[t_0, t_1] \times \mathbb{S}^2} |\mathring{\eta}|^{1/2} \mathring{\eta}^{ab} \left( (\mathrm{d}\xi_i)_a (\mathrm{d}\xi_i)_b - (\mathrm{d} \log \sin t)_a (\mathrm{d}\xi_i)_b \xi_i \right. \\ & \left. + \frac{1}{4} (\mathrm{d} \log \sin t)_a (\mathrm{d} \log \sin t)_b \xi_i^2 \right). \end{aligned} \quad (4.7)$$

We will follow now the method used in the preceding sections for the original  $\phi$  fields. Some details will be omitted owing to the similarity with the previous derivations. Let us consider then the space  $\mathcal{S}_\xi$  of smooth and symmetric real solutions to equation (4.6) and expand  $\xi \in \mathcal{S}_\xi$  as

$$\xi(t, s) = \sum_{\ell=0}^{\infty} \left( b_\ell z_\ell(t) Y_{\ell 0}(s) + \overline{b_\ell z_\ell(t) Y_{\ell 0}(s)} \right), \quad (4.8)$$

where  $z_\ell(t)$  are complex functions satisfying the equations

$$\ddot{z}_\ell + \left( \frac{1}{4} (1 + \csc^2 t) + \ell(\ell + 1) \right) z_\ell = 0. \quad (4.9)$$

The functions  $z_\ell$  can be easily written in terms the functions  $y_\ell$  appearing in (2.7) and satisfying (2.11)

$$z_\ell(t) = \sqrt{\sin t} y_\ell(t).$$

We immediately find that the Wronskian is now normalized to be

$$z_\ell \dot{\bar{z}}_\ell - \bar{z}_\ell \dot{z}_\ell = i. \quad (4.10)$$

This allows us to write the symplectic structure in  $\mathcal{S}_\xi$ , derived from (4.7), as

$$\Omega_\xi(\xi_1, \xi_2) = \int_{\mathbb{S}^2} |\gamma|^{1/2} \iota_t^* \left( \xi_2 \dot{\xi}_1 - \xi_1 \dot{\xi}_2 \right) = i \sum_{\ell=0}^{\infty} (\bar{b}_{1\ell} b_{2\ell} - \bar{b}_{2\ell} b_{1\ell}), \quad \forall \xi_1, \xi_2 \in \mathcal{S}_\xi.$$

## Classical evolution

We can consider now the classical functional time evolution operator  $\mathcal{T}_{(t_0, t_1)} : \Gamma_\xi \rightarrow \Gamma_\xi$  in the covariant phase space  $\Gamma_\xi = (\mathcal{S}_\xi, \Omega_\xi)$ . As before, we will write it in the form

$$(\mathcal{T}_{(t_0, t_1)} \xi)(t, s) := \sum_{\ell=0}^{\infty} \left( \mathbf{b}_\ell(t_0, t_1) z_\ell(t) Y_{\ell 0}(s) + \overline{\mathbf{b}_\ell(t_0, t_1) z_\ell(t) Y_{\ell 0}(s)} \right). \quad (4.11)$$



In this case, the map  $\mathcal{T}_{(t_0, t_1)} = \mathfrak{J}_{t_0} \circ \mathfrak{J}_{t_1}^{-1}$  is constructed from

$$\mathfrak{J}_{t_1}^{-1} : \Gamma_\xi \rightarrow \Upsilon, \quad \xi \mapsto (Q, P) = \mathfrak{J}_{t_1}^{-1}(\xi), \quad (4.12)$$

defined by<sup>11</sup>

$$Q(s) := \xi(t_1, s) = \sum_{\ell=0}^{\infty} \left( b_\ell z_\ell(t_1) Y_{\ell 0}(s) + \overline{b_\ell z_\ell(t_1) Y_{\ell 0}(s)} \right), \quad (4.13)$$

$$\begin{aligned} P(s) &:= \dot{\xi}(t_1, s) - \frac{1}{2} \cot t_1 \xi(t_1, s) \\ &= \sum_{\ell=0}^{\infty} \left( b_\ell [\dot{z}_\ell(t_1) - \frac{1}{2} \cot t_1 z_\ell(t_1)] Y_{\ell 0}(s) + \overline{b_\ell [\dot{z}_\ell(t_1) - \frac{1}{2} \cot t_1 z_\ell(t_1)] Y_{\ell 0}(s)} \right), \end{aligned} \quad (4.14)$$

and from

$$\mathfrak{J}_{t_0} : \Upsilon \rightarrow \Gamma_\xi, \quad (Q, P) \mapsto \xi = \mathfrak{J}_{t_0}(Q, P) \quad (4.15)$$

defined, in terms of the Fourier coefficients  $b_\ell$  of  $\xi$  (4.8), by

$$b_\ell(t_0) = -i \left[ \dot{\bar{z}}_\ell(t_0) - \frac{1}{2} \cot t_0 \bar{z}_\ell(t_0) \right] \int_{\mathbb{S}^2} |\gamma|^{1/2} Y_{\ell 0} Q + i \bar{z}_\ell(t_0) \int_{\mathbb{S}^2} |\gamma|^{1/2} Y_{\ell 0} P.$$

From these expressions we obtain

$$\begin{aligned} \mathfrak{b}_\ell(t_0, t_1) &= -i \left[ z_\ell(t_1) \left( \dot{\bar{z}}_\ell(t_0) - \frac{1}{2} \cot t_0 \bar{z}_\ell(t_0) \right) - \bar{z}_\ell(t_0) \left( \dot{z}_\ell(t_1) - \frac{1}{2} \cot t_1 z_\ell(t_1) \right) \right] b_\ell \\ &\quad -i \left[ \bar{z}_\ell(t_1) \left( \dot{z}_\ell(t_0) - \frac{1}{2} \cot t_0 z_\ell(t_0) \right) - z_\ell(t_0) \left( \dot{\bar{z}}_\ell(t_1) - \frac{1}{2} \cot t_1 \bar{z}_\ell(t_1) \right) \right] \bar{b}_\ell. \end{aligned} \quad (4.16)$$

## Quantum evolution

We will analyze the continuity of the symplectic transformation defined by (4.11) and (4.16) at the end of this section and consider first the unitarity condition for the quantum evolution in the corresponding Fock space quantization

$$\sum_{\ell=0}^{\infty} |\beta_\ell^\xi(t_0, t_1 | z_\ell)|^2 = \sum_{\ell=0}^{\infty} \left( \text{Re}^2(\beta_\ell^\xi(t_0, t_1 | z_\ell)) + \text{Im}^2(\beta_\ell^\xi(t_0, t_1 | z_\ell)) \right) < \infty, \quad (4.17)$$

for all  $t_0, t_1 \in (0, \pi)$ , where

$$\beta_\ell^\xi(t_0, t_1 | z_\ell) := z_\ell(t_1) \left( \dot{z}_\ell(t_0) - \frac{1}{2} \cot t_0 z_\ell(t_0) \right) - z_\ell(t_0) \left( \dot{z}_\ell(t_1) - \frac{1}{2} \cot t_1 z_\ell(t_1) \right). \quad (4.18)$$

The general solution of equation (4.9) with the normalization (4.10) can be written, again,

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<sup>11</sup> Notice that the space of Cauchy data for the  $\xi$ -field equations is also  $\Upsilon$ .

in terms of associated Legendre functions (2.16) in the form

$$\begin{aligned} z_\ell(t) &= \rho_\ell \sqrt{\sin t} u_{0\ell}(t) + (\nu_\ell + i\rho_\ell^{-1}) \sqrt{\sin t} v_{0\ell}(t) \\ &= \rho_\ell \tilde{u}_{0\ell}(t) + (\nu_\ell + i\rho_\ell^{-1}) \tilde{v}_{0\ell}(t), \end{aligned}$$

where, as above,  $\rho_\ell > 0$  and  $\nu_\ell \in \mathbb{R}$  parameterize different choices of one-particle Hilbert spaces, and we have defined  $\tilde{u}_{0\ell} := \sqrt{\sin t} u_{0\ell}$  and  $\tilde{v}_{0\ell} := \sqrt{\sin t} v_{0\ell}$ . We have to discuss now the convergence condition expressed in (4.17). Let us consider first

$$\text{Im}(\beta_\ell^\xi(t_0, t_1 | z_\ell)) = \tilde{A}_\ell(t_0, t_1) + 2\nu_\ell \rho_\ell^{-1} \tilde{B}_\ell(t_0, t_1)$$

where

$$\begin{aligned} \tilde{A}_\ell(t_0, t_1) &:= \tilde{u}_{0\ell}(t_1) \dot{\tilde{v}}_{0\ell}(t_0) - \tilde{u}_{0\ell}(t_0) \dot{\tilde{v}}_{0\ell}(t_1) + \dot{\tilde{u}}_{0\ell}(t_0) \tilde{v}_{0\ell}(t_1) - \dot{\tilde{u}}_{0\ell}(t_1) \tilde{v}_{0\ell}(t_0) \\ &\quad + \frac{1}{2}(\cot t_1 - \cot t_0) (\tilde{u}_{0\ell}(t_1) \tilde{v}_{0\ell}(t_0) + \tilde{u}_{0\ell}(t_0) \tilde{v}_{0\ell}(t_1)), \\ \tilde{B}_\ell(t_0, t_1) &:= \tilde{v}_{0\ell}(t_1) \dot{\tilde{v}}_{0\ell}(t_0) - \tilde{v}_{0\ell}(t_0) \dot{\tilde{v}}_{0\ell}(t_1) + \frac{1}{2}(\cot t_1 - \cot t_0) \tilde{v}_{0\ell}(t_0) \tilde{v}_{0\ell}(t_1). \end{aligned}$$

The asymptotic behavior of  $\tilde{A}_\ell$  and  $\tilde{B}_\ell$  as  $\ell \rightarrow \infty$  can be obtained from (4.5) and (2.16)

$$\tilde{A}_\ell(t_0, t_1) \sim \frac{1}{4\ell} (\cot t_1 - \cot t_0) \cos[(\ell + 1/2)(t_0 + t_1)], \quad (4.19)$$

$$\tilde{B}_\ell(t_0, t_1) \sim \frac{\pi}{4} \sin[(\ell + 1/2)(t_1 - t_0)]. \quad (4.20)$$

We then conclude that  $\text{Im}(\beta_\ell^\xi(t_0, t_1 | z_\ell))$  is square summable if  $\nu_\ell \rho_\ell^{-1} \sim \ell^{-a}$  (with  $a \geq 1$ ) when  $\ell \rightarrow \infty$ . For the real part we have

$$\text{Re}(\beta_\ell^\xi(t_0, t_1 | z_\ell)) = \rho_\ell \nu_\ell \tilde{A}_\ell(t_0, t_1) + (\nu_\ell^2 - \rho_\ell^{-2}) \tilde{B}_\ell(t_0, t_1) + \rho_\ell^2 \tilde{C}_\ell(t_0, t_1),$$

where

$$\begin{aligned} \tilde{C}_\ell(t_0, t_1) &:= \tilde{u}_{0\ell}(t_1) \dot{\tilde{u}}_{0\ell}(t_0) - \tilde{u}_{0\ell}(t_0) \dot{\tilde{u}}_{0\ell}(t_1) + \frac{1}{2}(\cot t_1 - \cot t_0) \tilde{u}_{0\ell}(t_0) \tilde{u}_{0\ell}(t_1) \\ &\sim \frac{1}{\pi} \sin[(\ell + 1/2)(t_1 - t_0)], \quad \text{when } \ell \rightarrow \infty. \end{aligned} \quad (4.21)$$

The asymptotic behavior as  $\ell \rightarrow \infty$  of  $\text{Re}(\beta_\ell^\xi(t_0, t_1 | z_\ell))$  can be obtained now from (4.19), (4.20), and (4.21). If we choose now  $\rho_\ell$  in such a way that

$$\rho_\ell \sim \sqrt{\frac{\pi}{2}} \quad \text{and} \quad \nu_\ell \sim \ell^{-a} \quad (a \geq 1) \quad \text{as } \ell \rightarrow \infty \quad (4.22)$$

we also guarantee that  $\text{Re}(\beta_\ell^\xi(t_0, t_1 | z_\ell))$  is square summable, and hence  $\beta_\ell^\xi(t_0, t_1 | z_\ell)$ .

We end this section by showing that the linear symplectic map  $\mathcal{T}_{(t_0, t_1)}$  is continuous in the norm  $\|\cdot\| = \sqrt{\langle \cdot | \cdot \rangle_{\mathcal{P}}}$  associated to the inner product (3.3) for the complex structures characterized by the pairs  $(\rho_\ell, \nu_\ell)$  verifying (4.22). That is, there exists some  $K(t_0, t_1) > 0$

such that

$$\|\kappa(\mathcal{T}_{(t_0, t_1)}\xi)\| \leq K(t_0, t_1)\|\kappa(\xi)\|$$

for all  $\xi \in \mathcal{S}_\xi$ , where  $\kappa : \mathcal{S}_{\xi_{\mathbb{C}}} \rightarrow \mathcal{P}_\xi$  is the  $\mathbb{C}$ -linear projector defined by the splitting  $\mathcal{S}_{\xi_{\mathbb{C}}} = \mathcal{P}_\xi \oplus \bar{\mathcal{P}}_\xi$ . By using (4.11) and (4.16) it is straightforward to show that

$$\|\kappa(\mathcal{T}_{(t_0, t_1)}\xi)\|^2 = \sum_{\ell=0}^{\infty} |\mathbf{b}_\ell(t_0, t_1)|^2 \leq \sum_{\ell=0}^{\infty} \left( |\alpha_\ell(t_0, t_1|z_\ell)|^2 + |\beta_\ell^\xi(t_0, t_1|z_\ell)|^2 \right) |b_\ell|^2 \quad (4.23)$$

where

$$\alpha_\ell^\xi(t_0, t_1|z_\ell) := z_\ell(t_1) \left( \dot{z}_\ell(t_0) - \frac{1}{2} \cot t_0 \bar{z}_\ell(t_0) \right) - \bar{z}_\ell(t_0) \left( \dot{z}_\ell(t_1) - \frac{1}{2} \cot t_1 z_\ell(t_1) \right),$$

and  $\beta_\ell^\xi(t_0, t_1|z_\ell)$  is given by (4.18). We have shown above that the sequence  $\{|\beta_\ell^\xi(t_0, t_1|z_\ell)|^2\}_{\ell=0}^\infty$  is bounded (actually square summable) so if we can see now that  $\{|\alpha_\ell^\xi(t_0, t_1|z_\ell)|^2\}_{\ell=0}^\infty$  is also a bounded sequence the continuity of  $\mathcal{T}_{(t_0, t_1)}$  follows directly from equation (4.23). By expanding  $z_\ell = \rho_\ell \tilde{u}_{0\ell} + (\nu_\ell + i\rho_\ell^{-1})\tilde{v}_{0\ell}$ —and making use of (4.6), (4.19), (4.20), (4.21), and (4.22)—it is possible to show that

$$\begin{aligned} \operatorname{Re}(\alpha_\ell^\xi(t_0, t_1|z_\ell)) &= \rho_\ell^2 \tilde{C}_\ell(t_0, t_1) + \rho_\ell \nu_\ell \tilde{A}_\ell(t_0, t_1) + (\nu_\ell^2 + \rho_\ell^{-2}) \tilde{B}_\ell(t_0, t_1) \\ &\sim \sin[(\ell + 1/2)(t_1 - t_0)] \quad \text{when } \ell \rightarrow \infty, \\ \operatorname{Im}(\alpha_\ell^\xi(t_0, t_1|z_\ell)) &= \tilde{v}_{0\ell}(t_1) \dot{\tilde{u}}_{0\ell}(t_0) - \tilde{u}_{0\ell}(t_0) \dot{\tilde{v}}_{0\ell}(t_1) + \tilde{v}_{0\ell}(t_0) \dot{\tilde{u}}_{0\ell}(t_1) - \tilde{u}_{0\ell}(t_1) \dot{\tilde{v}}_{0\ell}(t_0) \\ &\quad + \frac{1}{2}(\cot t_0 - \cot t_1) \left( \tilde{u}_{0\ell}(t_1) \tilde{v}_{0\ell}(t_0) - \tilde{u}_{0\ell}(t_0) \tilde{v}_{0\ell}(t_1) \right) \\ &\sim \cos[(\ell + 1/2)(t_1 - t_0)] \quad \text{when } \ell \rightarrow \infty. \end{aligned}$$

From these equations it is clear that there exists a  $K^2(t_0, t_1) > 0$  such that

$$|\alpha_\ell^\xi(t_0, t_1|z_\ell)|^2 + |\beta_\ell^\xi(t_0, t_1|z_\ell)|^2 \leq K^2(t_0, t_1), \quad \forall \ell \in \mathbb{N} \cup \{0\}.$$

Then, using (4.23), we get that  $\|\kappa(\mathcal{T}_{(t_0, t_1)}\xi)\|^2 \leq K^2(t_0, t_1)\|\kappa(\xi)\|^2$ , and hence  $\mathcal{T}_{(t_0, t_1)}$  is continuous. In conclusion, by imposing suitable conditions (4.22) on the parameters  $\rho_\ell$  and  $\nu_\ell$ , it is possible to find  $SO(3)$ -complex structures (and, hence, subspaces  $\mathcal{P}$ ) such that the quantum dynamics can be unitarily implemented in  $\mathcal{F}_s(\mathcal{H}_\mathcal{P})$ .

## B. Uniqueness of the Fock quantization

We will show in this section that any two Fock quantizations of the field  $\xi$  corresponding to  $SO(3)$ -invariant complex structures, for which the dynamics can be unitarily implemented, are equivalent. To this end, let us recall some properties of the  $SO(3)$ -invariant complex structures considered in section III B. Given any invariant complex structure  $J$ , it is possible to characterize its action on the fixed basis  $\phi_{0\ell}$  that defines the complex structure  $J_0$ . This action is given by equation (3.7). As we can see there exists a linear symplectic

transformation  $T_J$  connecting them, so that  $J = T_J \circ J_0 \circ T_J^{-1}$ . Explicitly

$$T_J = \bigoplus_{\ell=0}^{\infty} \begin{pmatrix} (\tau_1^\ell)_J I_{11}^\ell & (\tau_2^\ell)_J I_{12}^\ell \\ (\bar{\tau}_2^\ell)_J I_{21}^\ell & (\bar{\tau}_1^\ell)_J I_{22}^\ell \end{pmatrix}, \quad (4.24)$$

with

$$\begin{aligned} (\tau_1^\ell)_J &:= \sqrt{(1 + |J_{11}^\ell|)/2} \quad (\text{up to multiplicative phase}), \\ (\tau_2^\ell)_J &:= \frac{iJ_{12}^\ell}{2(\tau_1^\ell)_J}. \end{aligned}$$

Notice that  $J_0$ , defined by the set of functions  $\{z_{0\ell}(t) = \tilde{u}_{0\ell}(t) + i\tilde{v}_{0\ell}\}_{\ell=0}^{\infty}$ , corresponding to  $\rho_\ell = 1$  and  $\nu_\ell = 0$ , does not lead to a unitary implementation of dynamics. In this context, it is fixed just to compare different complex structures. Let us consider then any two  $SO(3)$ -invariant complex structures,  $J$  and  $J'$ , for which the dynamics is unitary. They will define unitarily equivalent quantum theories if and only if the linear symplectic transformation  $T_{J,J'} := T_J \circ T_{J'}^{-1}$  connecting them through  $J = T_{J,J'} \circ J' \circ T_{J,J'}^{-1}$  is unitarily implementable. This is the case if the sequence

$$\{(\tau_2^\ell)_J (\tau_1^\ell)_{J'} - (\tau_1^\ell)_J (\tau_2^\ell)_{J'}\}_{\ell=0}^{\infty}$$

is square summable. Taking into account the relations (3.8) and (3.9), as well as the asymptotic behaviors (4.22), the previous condition is indeed verified, so the quantum theories defined by  $J$  and  $J'$  are unitarily equivalent.

### C. Normalizability of the action of the Hamiltonian on the vacuum state

We discuss here an interesting feature of the quantum dynamics for these systems: The fact that, even though the evolution is unitarily implemented, the time-dependent quantum Hamiltonian, constructed from the classical one by following the standard rules of quantization, has the striking property that Fock space vectors corresponding to a finite number of particle-like excitations do not belong to its domain. This also happens in the  $\mathbb{T}^3$  case [13].

The classical Hamiltonian on the canonical phase space  $\Upsilon$  in the  $\xi$ -description of the system is derived from the action (4.7). It is given by

$$H(Q, P; t) = \frac{1}{2} \int_{\mathbb{S}^2} |\gamma|^{1/2} (P^2 + \cot t PQ - Q \Delta_{\mathbb{S}^2} Q). \quad (4.25)$$

Notice that the time-dependent (non-autonomous) Hamiltonian (4.25) is an *indefinite* quadratic form with a cross term involving  $Q$  and  $P$ . Let us discuss now the quantum Hamiltonian. To this end we first write the formal quantum version of (4.13) and (4.14)

that should be understood as operator-valuated distributions on  $\mathbb{S}^2$  for each value of  $t$

$$\begin{aligned}\hat{Q}(t, s) &:= \sum_{\ell=0}^{\infty} \left( z_{\ell}(t) Y_{\ell 0}(s) \hat{b}_{\ell} + \overline{z_{\ell}(t) Y_{\ell 0}(s)} \hat{b}_{\ell}^{\dagger} \right), \\ \hat{P}(t, s) &:= \sum_{\ell=0}^{\infty} \left( \left[ \dot{z}_{\ell}(t) - \frac{1}{2} \cot t z_{\ell}(t) \right] Y_{\ell 0}(s) \hat{b}_{\ell} + \overline{\left[ \dot{z}_{\ell}(t) - \frac{1}{2} \cot t z_{\ell}(t) \right] Y_{\ell 0}(s)} \hat{b}_{\ell}^{\dagger} \right),\end{aligned}$$

where  $\hat{b}_{\ell}$  and  $\hat{b}_{\ell}^{\dagger}$  are the annihilation and creation operators associated to the modes  $\xi_{\ell} = z_{\ell} Y_{\ell 0}$ , respectively. Substituting these expressions in (4.25), and after normal ordering, we find

$$\hat{H}(t) = \frac{1}{2} \sum_{\ell=0}^{\infty} \left( K_{\ell}(t) \hat{b}_{\ell}^2 + \bar{K}_{\ell}(t) \hat{b}_{\ell}^{\dagger 2} + 2G_{\ell}(t) \hat{b}_{\ell}^{\dagger} \hat{b}_{\ell} \right), \quad (4.26)$$

where

$$\begin{aligned}K_{\ell}(t) &:= \left( \dot{z}_{\ell}(t) - \frac{1}{2} \cot t z_{\ell}(t) \right)^2 + \ell(\ell+1) z_{\ell}^2(t) + \cot t \left( \dot{z}_{\ell}(t) - \frac{1}{2} \cot t z_{\ell}(t) \right) z_{\ell}(t), \\ G_{\ell}(t) &:= \left| \dot{z}_{\ell}(t) - \frac{1}{2} \cot t z_{\ell}(t) \right|^2 + \ell(\ell+1) |z_{\ell}(t)|^2 \\ &\quad + \frac{1}{2} \cot t \left( \left( \dot{z}_{\ell}(t) - \frac{1}{2} \cot t z_{\ell}(t) \right) \bar{z}_{\ell}(t) + \left( \dot{\bar{z}}_{\ell}(t) - \frac{1}{2} \cot t \bar{z}_{\ell}(t) \right) z_{\ell}(t) \right).\end{aligned} \quad (4.27)$$

The action of the quantum Hamiltonian on the vacuum  $|0\rangle$  is now

$$\hat{H}(t)|0\rangle = \frac{1}{\sqrt{2}} \sum_{\ell=0}^{\infty} \bar{K}_{\ell}(t) |2_{\ell}\rangle,$$

where  $\sqrt{2}|2_{\ell}\rangle = \hat{b}_{\ell}^{\dagger 2} |0\rangle$ . The state  $\hat{H}(t)|0\rangle$  will be normalizable if and only if

$$\sum_{\ell=0}^{\infty} |K_{\ell}(t)|^2 < \infty. \quad (4.28)$$

Taking into account the asymptotic behaviors of the Legendre functions (4.5) when  $\ell \rightarrow \infty$ , and imposing the conditions  $\rho_{\ell} \sim \sqrt{\pi/2}$  and  $\nu_{\ell} \sim \ell^{-a}$  discussed above to guarantee the unitary implementation of the time evolution, we get

$$\begin{aligned}z_{\ell}(t) &= \rho_{\ell} \sqrt{\sin t} u_{0\ell}(t) + (\nu_{\ell} + i\rho_{\ell}^{-1}) \sqrt{\sin t} v_{0\ell}(t) \sim \frac{1}{\sqrt{2\ell}} \exp \left( -i[(\ell+1/2)t - \pi/4] \right), \\ \dot{z}_{\ell}(t) - \frac{1}{2} \cot t z_{\ell}(t) &\sim -i \sqrt{\frac{\ell}{2}} \exp \left( -i[(\ell+1/2)t - \pi/4] \right).\end{aligned}$$

It is straightforward now to compute the asymptotic behavior of  $K_{\ell}(t)$  defined in (4.27) and also check that condition (4.28) is not satisfied. We then conclude that the Fock space vacuum does not belong to the domain of the Hamiltonian for any time  $t \in (0, \pi)$  and, hence, the action of the Hamiltonian on  $n$ -particle states is not defined either.

It is important to point out that it is possible to consider the definition of the quantum Hamiltonian in a more mathematical framework. It is well known that the unitary evolution operator  $\hat{U}(t_0, t_1)$  can be derived from the evolution of creation operators in the Heisenberg picture and the evolution of the vacuum state. Furthermore, the vacuum evolution can be written in closed form as in [22, 23] and is given by a completely analogous formula. As expected in a non-autonomous system, the vacuum state (and, hence,  $n$ -particle states) is not stable under time evolution. After computing the explicit form of the evolution operator, it is possible to study the differentiability of  $\hat{U}(t_0, t_1)$  in a rigorous mathematical sense and then, whenever  $\hat{U}$  is differentiable, we can define the quantum Hamiltonian of the system. This is beyond the scope of the present paper.

We end this section by noting that the covariant phase space  $\Gamma_\xi$  defined by (4.7) can be equivalently derived from the simpler action

$$s_0(\xi) = -\frac{1}{2} \int_{[t_0, t_1] \times \mathbb{S}^2} |\dot{\eta}|^{1/2} \dot{\eta}^{ab} \left( (d\xi)_a (d\xi)_b + \frac{1}{4} (1 + \csc^2 t) \xi^2 \right). \quad (4.29)$$

This variational principle gives a time-dependent, *positive definite*, diagonal Hamiltonian of the form

$$H_0(Q, P; t) = \frac{1}{2} \int_{\mathbb{S}^2} |\gamma|^{1/2} \left( P^2 + Q \left[ \frac{1}{4} (1 + \csc^2 t) - \Delta_{\mathbb{S}^2} \right] Q \right).$$

There are no subtleties associated to the domain of the quantum counterpart of  $H_0$  in the sense that now the Fock space vacuum belongs to the domain of the Hamiltonian. The action principle (4.29) is related to the Einstein-Hilbert action for the Gowdy models (2.2) through a field redefinition. In fact both actions can be connected by a time-dependent canonical transformation though nothing guarantees that this can be unitarily implemented, in which case the quantizations would be different.

## V. CONCLUSIONS AND COMMENTS

As we have shown in the paper there is a very natural framework to discuss issues related to the unitary implementability of dynamics in the compact Gowdy models. The key idea is to use a covariant phase space approach where the solutions to the field equations play the main role. The best way to describe these solution spaces [17] is by rewriting the field equations in terms of certain auxiliary background metrics that are conformally equivalent to some specially simple and natural ones. For the  $\mathbb{T}^3$  case, this metric is the flat metric on  $(0, \infty) \times \mathbb{T}^2$ , and for the  $\mathbb{S}^1 \times \mathbb{S}^2$  and  $\mathbb{S}^3$  examples the metric is the Einstein metric on  $(0, \pi) \times \mathbb{S}^2$ . It is important to highlight the fact that this is possible as a consequence of the symmetry left in the model after its reduction to  $(1+2)$ -dimensions. This symmetry is generated by the Killing field remaining after the Geroch reduction from  $(1+3)$  to  $(1+2)$  dimensions. An advantage of this approach is the fact that the time singularities of the metric are completely described by the time-dependent conformal factors. The metric becomes singular whenever they cancel. This ultimately explains why a simple field redefinition involving *precisely* these conformal factors suffices to cure the problems associated with the quantum unitary evolution. In fact a conformal transformation defined with the help of these conformal factors shifts the singularity of the metric to one appearing in a time-dependent potential

term that becomes singular when the full metric does.

A first result of the paper is a proof of the fact that the impossibility to get unitary dynamics in terms of the original variables that naturally appear in the description of the model is insensitive to the choice of the complex structure used in the quantization. This result generalizes the conclusion reached in [13] for the  $\mathbb{T}^3$  case to the topologies considered here ( $\mathbb{S}^1 \times \mathbb{S}^2$  and  $\mathbb{S}^3$ ). The starting point of the approach that we develop in the paper is to consider the possibility of achieving unitary quantum evolution by making an appropriate choice of complex structure; only when this fails are we forced to introduce new variables to describe the system<sup>12</sup>. It is interesting to point out in this respect that the type of unitarity problem discussed here cannot always be fixed by time-dependent redefinitions of the type used in the paper; in fact it is possible to give examples (a massless scalar field evolving in a de Sitter background) where this is not the case [24]. The ultimate reason why the method used here does not work in these other models is the fact that the time-dependent potential written in terms of the new fields is not as well behaved as the ones that show up in the treatment of the Gowdy models.

A second point that we want to comment on is the uniqueness issue. In the case of the  $\mathbb{T}^3$  Gowdy models the presence of a constraint, and the corresponding symmetry generated by it, gives the possibility of introducing a physically sensible criterion to select the complex structure: invariance under this symmetry [14]. This is not the case for the other compact topologies that we consider here for which, as we showed in [17], there are no extra constraints after deparameterization. It is important to realize in this respect that we have used the  $SO(3)$  symmetry associated to the background metric to select a preferred class of complex structures.

Notice that at this point we still have many different  $SO(3)$ -invariant Fock quantizations  $\mathcal{F}_s(\mathcal{H}_{\mathcal{P}})$  labeled by  $\mathcal{P}$  that, in principle, are not guaranteed to be equivalent. In such a situation we would need an additional criterion to pick one. Once we require that the quantum dynamics is unitary we find that all of them are unitarily equivalent.

A final comment is to note that the same scheme followed here works in the  $\mathbb{T}^3$  case (with or without massless scalar matter). For the vacuum case one directly recovers several interesting results discussed in the literature for this system.

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<sup>12</sup> For the  $\mathbb{T}^3$  case this is obtained as a corollary of the uniqueness result described in [16].

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